Lecture 1

1 Sets

1.1 Sets and Subsets

A **set** is a collection of distinct objects satisfying certain properties. The objects in the collection are called **elements** or **members**.

Given a set A; we write "$x \in A$" to say that x is an element of A or x belongs to A; and write "$x \notin A$" to say that x is not an element of A or x does not belong to A. Sets are usually denoted by uppercase letters such that $A, B, C, \ldots$; elements of sets are usually denoted by lowercase letters such that $a, b, c, \ldots$.

There are two ways to express a set. One is to list all elements of the set; the other one is to point out the attributes for the elements of the set. For instance,

$$A = \{1, -1\}; \quad B = \{x \mid x \text{ real number}, \ x^2 = 1\}.$$

A set A is called a **subset** of a set B, written $A \subseteq B$, if every element of A is an element of B. Two sets A and B are said to be **equal**, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

The set without any element is call the **empty set**, denoted $\emptyset$. The empty set may be exhibit as $\emptyset = \{\ \}$. There is only one empty set. The empty set $\emptyset$ is a subset of any set.

We constantly use subsets of the following sets in the course.

$$\begin{aligned}
\mathbb{P} : &= \text{the set of all positive integers.} \\
\mathbb{N} : &= \text{the set of all nonnegative integers.} \\
\mathbb{Z} : &= \text{the set of all integers.} \\
\mathbb{Q} : &= \text{the set of all rational numbers.} \\
\mathbb{R} : &= \text{the set of all real numbers.} \\
\mathbb{C} : &= \text{the set of all complex numbers.}
\end{aligned}$$

1.2 Intersection and Union

Let A and B be sets. The **intersection** of A and B, written $A \cap B$, is the set consisting of all elements which belongs to both sets, i.e.,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The union of A and B, written $A \cup B$, is the set consisting of all elements that belong to either A or B, i.e.,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The intersection and union of finite number of sets $A_1, A_2, \ldots, A_n$ are defined respectively as follows:

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i,$$

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i.$$

Similarly, for infinitely many sets $A_1, A_2, \ldots$, their intersection and uion are defined as

$$\bigcap_{i=1}^{\infty} A_i = \{x \mid x \in A_i \text{ for all } i\},$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid x \in A_i \text{ for some } i\}.$$

1.3 Difference

Let A and B be two sets. The **difference** from A to B is the set

$$A - B := \{x \mid x \in A \text{ and } x \notin B\};$$

it is also called the **relative complement** of B in A.

1. If $A = \{x \mid x \in \mathbb{R},\ 0 \leq x \leq 3\}$, $B = \{x \mid x \in \mathbb{R},\ 1 \leq x < 2\}$. Then

$$A - B = \{x \mid x \in \mathbb{R},\ 0 \leq x < 1\} \cup \{x \mid x \in \mathbb{R},\ 2 \leq x \leq 3\}.$$

2. If $A = \{x \mid x \in \mathbb{R},\ 0 \leq x \leq 2\}$, $B = \{x \mid x \in \mathbb{Q},\ 1 < x \leq 3\}$, then

$$A - B = \{x \mid x \in \mathbb{R},\ 0 \leq x \leq 1\} \cup \{x \mid x \text{ irrational},\ 1 < x < 2\}.$$

When we only consider subsets of a particular set U, we sometimes refer the set U a **universal set**. (A universal set is not universal, not including everything.) If so, we refer the relative complement $U - A$ for a subset $A \subset U$ to just **complement** and is denoted by A^c.

Let A and B be sets. We have the **DeMorgan Law**:

$$(A^c)^c = A, \quad (A \cap B)^c = A^c \cup B^c, \quad (A \cup B)^c = A^c \cap B^c.$$

2 Power Set

The **power set** $\mathcal{P}(A)$ of a set A is the set of all subsets of A, that is,

$$\mathcal{P}(A) := \{S \mid S \subseteq A\}.$$

If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$, which is not empty and contains exactly one element; this unique element is the empty set $\emptyset$. If $A = \{a\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$. If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. If $A = \{a, b, c\}$, then

$$\mathcal{P}(A) = \big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\big\}.$$

3 Product

The **Cartesian product** (or just **product**) of two sets A and B is the set

$$A \times B := \big\{(a, b) \mid a \in A \text{ and } b \in B\big\}.$$

The product of a finite family $A_1, A_2, \ldots, A_n$ of sets is the set

$$A_1 \times A_2 \times \cdots \times A_n := \big\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\big\}.$$

The product of a finite family $A_1, A_2, \ldots, A_n$ of sets is the set

$$\begin{aligned} \prod_{i=1}^{n} A_i &= A_1 \times A_2 \times \cdots \times A_n \\ &= : \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}. \end{aligned}$$

4 Finite Sets

Proposition 1. *If A and B are finite sets, then*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof. Let $A \cap B = \{x_1, \ldots, x_k\}$. Then $|A \cap B| = k$. We may write

$$A = \{x_1, \ldots, x_k, a_1, \ldots, a_l\},$$

$$B = \{x_1, \ldots, x_k, b_1, \ldots, x_m\}.$$

Then $|A| = k + l$, $|B| = k + m$, and

$$A \cup B = \{x_1, \ldots, x_k, a_1, \ldots, a_l, b_1, \ldots, b_m\}.$$

Thus

$$\begin{aligned} |A \cup B| &= k + l + m \\ &= (k + l) + (k + m) - k \\ &= |A| + |B| - |A \cap B|. \end{aligned}$$

□

Proposition 2. *If A, B, and C are finite sets, then*

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + A \cap B \cap C|. \end{aligned} \tag{1}$$

Proof. For the finite set A and $B \cup C$, applying Proposition 1, we have

$$|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|. \tag{2}$$

Since $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, applying Proposition 1 again, we have

$$|B \cup C| = |B| + |C| - |B \cap C|, \tag{3}$$

$$|A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap A \cap C|, \tag{4}$$

Notice that $A \cap B \cap A \cap C = A \cap B \cap C$. Combine (2)–(4); we obtain (1). □

Key Words: Subsets, intersection, union, relative complement, Venn diagram, Cartesian product, cardinality, inclusion-exclusion.

Lecture 2

5 Proofs

5.1 Statements and Implication

A **statement** is a sentence that is either true or false, but not both.

(1) The squares of odd integer is odd.
(2) No real number has square equal to -1.
(3) Every positive integer is equal to a sum of two integer squares.

The sentence such as "How are you?" is not a statement.

Let P and Q be two statements. We say that P **implies** Q, written $P \Rightarrow Q$, if whenever P is true then Q is also true. The statement "$P \Rightarrow Q$" can be stated in three ways: If P then Q; Q if P; P only if Q.

Example 1. Let $P : x = 2$; $Q : x^2 < 6$. Then "$x = 2 \Rightarrow x^2 < 6$" can be stated in the following three ways.

$$\begin{aligned} \text{If } P \text{ then } Q\text{:} &\quad \text{If } x = 2 \text{ then } x^2 < 6. \\ Q \text{ if } P\text{:} &\quad x^2 < 6 \text{ if } x = 2. \\ P \text{ only if } Q\text{:} &\quad x = 2 \text{ only if } x^2 < 6. \end{aligned}$$

Example 2. Let P : It is raining; Q : The sky is cloudy. Then "It is raining $\Rightarrow$ The sky is cloudy" can be stated in three ways.

If P then Q: If it is raining then the sky is cloudy.
Q if P: The sky is cloudy if it is raining.
P only if Q: It is raining only if the sky is cloudy.

Implication: "It is raining $\Rightarrow$ The sky is cloudy."

Assumption	Deduction
It is raining	The sky is cloudy
It is not raining	No deduction possible
The sky is cloudy	No deduction possible
The sky is not cloudy	It is not raining

Two statements P and Q are said to be **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$, written $P \Leftrightarrow Q$. We also say "$P \Leftrightarrow Q$" as "P if and only if Q." For example, $x^2 = 2 \Leftrightarrow x^3 = 8$; John is black $\Leftrightarrow$ One of John's biological parent is black.

5.2 $\boldsymbol{P \Rightarrow Q}$ via $\boldsymbol{\bar{Q} \Rightarrow \bar{P}}$

The **negation** of a statement P is the opposite statement, "not P," written as $\bar{P}$. We demonstrate that $P \Rightarrow Q$ is equivalent to $\bar{Q} \Rightarrow \bar{P}$, which is called the **contrapositive form** of $P \Rightarrow Q$.

Example 3. Let $P : x = 2$; $Q : x^2 < 6$. We have

$$\begin{aligned} P \Rightarrow Q: &\quad x = 2 \Rightarrow x^2 < 6. \\ \bar{Q} \Rightarrow \bar{P}: &\quad x^2 \geq 6 \Rightarrow x \neq 2. \end{aligned}$$

Example 4. Let P : It is raining; Q : The sky is cloudy. Then

$P \Rightarrow Q$: If it is raining then the sky is cloudy.
$\bar{Q} \Rightarrow \bar{P}$: If the sky is sunshine then it is not raining.

5.3 Deduction

Example 5. (a) I am admitted to HKUST only if I am smart; (b) If I am smart then I do not need to work; (c) I have to work.
Answer. We write

H : I am admitted to HKUST.
S : I am smart.
W : I have to work.

Then (a) $H \Rightarrow S$, (b) $S \Rightarrow \bar{W}$, (c) W. Thus $W \Rightarrow \bar{S}$, $\bar{S} \Rightarrow \bar{H}$. So $\bar{H}$ is true, i.e., "I am not admitted to HKUST."

5.4 Direct Proof

Example 6. Prove that the square of odd integer is odd.

Proof. Let n be an odd integer. Then n is 1 more than an even integer, i.e., $n = 2k + 1$ for some integer k. Thus, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. This means that n^2 is 1 more than $4(k^2 + k)$, an even integer. Hence n^2 is odd.

$$n \text{ odd} \Rightarrow n = 2k + 1 \Rightarrow n^2 = 4(k^2 + k) + 1 \Rightarrow n^2 \text{ odd}.$$

□

5.5 Proof by Contradiction

Suppose we wish to prove a statement P. We first assume that P is false, that is, $\bar{P}$, then deduce a statement Q that is palpably false.

Example 7. Let n be an integer such that n^2 is a multiple of 3. Then n is also a multiple of 3.

Proof. Suppose n is not a multiple of 3. Then $n = 3k + 1$ or $n = 3k + 2$ for some integer k. In the case $n = 3k + 1$, we have

$$n^2 = (3k + 1) = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1,$$

which is not a multiple of 3. In the case $n = 3k + 2$, we have

$$n^2 = (3k + 2) = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1,$$

which is also not a multiple of 3. In both cases it is contradict to that n^2 is a multiple of 3. □

Lecture 3

An example of non-proof:

Example 8. Show $\sqrt{2} + \sqrt{6} < \sqrt{15}$.

Wrong proof:

$$\begin{aligned} \sqrt{2} + \sqrt{6} < \sqrt{15} &\Rightarrow (\sqrt{2} + \sqrt{6})^2 < 15 \\ &\Rightarrow 8 + 2\sqrt{12} < 15 \\ &\Rightarrow 2\sqrt{12} < 7 \\ &\Rightarrow 48 < 49. \end{aligned}$$

Proof. Suppose $\sqrt{2} + \sqrt{6} < \sqrt{15}$ is not true, i.e., $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$. Then

$$\begin{aligned} \sqrt{2} + \sqrt{6} \geq \sqrt{15} &\Rightarrow (\sqrt{2} + \sqrt{6})^2 \geq 15 \\ &\Rightarrow 8 + 2\sqrt{12} \geq 15 \\ &\Rightarrow 2\sqrt{12} \geq 7 \\ &\Rightarrow 48 \geq 49. \end{aligned}$$

The conclusion "$48 \geq 49$" is contradictory to "$48 < 49$." □

5.6 Disprove by Counterexample

Proving a statement to be false is called **disproving**.

Example 9. Every positive integer is a sum of two integer squares.

Answer: This is not true because 3 can not be written as a sum of two integer squares.

Sometimes it is difficult to prove directly a statement and is also difficult to find a counterexample. There are statements in mathematics (called open problems) that we know they are sure either true or false, but not both. However, we are just neither able to prove the statement nor to find a counterexample. For instance, the following is one of many famous open problems.

Example 10 (Goldbach Conjecture)**.** Every positive even integer larger than 2 can be written as a sum of two primes.

5.7 Another Example of Non-proof

Since $4 = 2 + 2$; $6 = 1 + 5$; $8 = 3 + 5$; $10 = 5 + 5$; $12 = 5 + 7$; $14 = 7 + 7$; $16 = 5 + 11$; $18 = 7 + 11$; $20 = 7 + 13$; $22 = 11 + 11$; $24 = 11 + 13$ are true, then the statement is true. This is typically argued by non-professional math fans. Of course, this is **not** a proof because we could not check all even integers.

Example 11. How many integers are there between 1000 and 9999 which contains the digits 0, 8, and 9 at least once?

Solution. Let $S = \{100, 1001, \ldots, 9999\}$. For $k = 0, 8, 9$, let A_k denote the subset of S, consisting of those integers that have no digit equal to k. Then the union $A_0 \cup A_8 \cup A_9$ is the set of integers in S which are missing either 0, 8, or 9. So the number we are asked for in question is

$$|S| - |A_0 \cup A_8 \cup A_9| = 9000 - |A_0 \cup A_8 \cup A_9|.$$

We therefore need to calculate $|A_0 \cup A_8 \cup A_9|$, which can be find by the inclusion-exclusion formula. Note that A_9 is the set of integers in S without digit 9; each integer in A_9 has four digits, the 1st digit has 8 choices, each of the other three digits has 9 choices. Then $|A_9| = 8 \cdot 9 \cdot 9 \cdot 9 = 5832$. Similarly,

$$|A_8| = 8 \cdot 9 \cdot 9 \cdot 9 = 5832, \quad |A_0| = 9 \cdot 9 \cdot 9 \cdot 9 = 6561,$$

$$|A_0 \cap A_8| = |A_0 \cap A_9| = 8 \cdot 8 \cdot 8 \cdot 8 = 4096, \quad |A_8 \cap A_9| = 7 \cdot 8 \cdot 8 \cdot 8 = 3584,$$

$$|A_0 \cap A_8 \cap A_9| = 7 \cdot 7 \cdot 7 \cdot 7 = 2401.$$

Thus

$$|A_0 \cup A_8 \cup A_9| = 5832 + 5832 + 6561 - 4096 - 4096 - 3584 + 2401 = 8850.$$

Therefore the answer is given by $9000 - 8850 = 150$.

Lecture 4

6 Counting

6.1 Permutation and Combination

Let $A_1, A_2, \ldots, A_k$ be finite sets. If $|A_1| = n_1$, $|A_2| = n_2$, …, $|A_k| = n_k$, it is easy to see the **Multiplication Rule**

$$|A_1 \times A_2 \times \cdots \times A_k| = n_1 n_2 \cdots n_k.$$

In particular, if $A_1 = A_2 = \cdots = A_k = A$ and $|A| = n$, we have

$$|A^k| = n^k.$$

A **word of length** k over A is an element of the form

$$a_1 a_2 \ldots a_k,$$

where $a_1, a_2, \ldots, a_k \in A$. A word of length k is also called a k**-arrangement** or k**-permutation** of A if the elements in the word are distinct. An n-permutation of A is just called a **permutation** of A.

Proposition 3. *Let A be a finite set with $|A| = n$. Then the number of k-permutations of A, denoted by $P(n, k)$, is given by*

$$P(n, k) = n(n-1)\cdots(n-k+1).$$

Proof. To form an arbitrary k-permutation $a_1 a_2 \cdots a_k$ of A, there are n choices for the 1st element a_1, $n-1$ choices for the 2nd element a_2, …, and $n-(k-1)$ choices for the kth (also last) element a_k. Hence, by the Multiplication Rule, the total number of possibilities is $n(n-1)\cdots(n-k+1)$. □

Corollary 4. *Let A be a finite set with $|A| = n$. Let*

$$P(A, k) = \{(a_1, a_2, \ldots, a_k) \in A^k \mid \text{all } a_1, a_2, \ldots, a_k \text{ are distinct}\}.$$

Then

$$|P(A, k)| = n(n-1)\cdots(n-k+1).$$

Let A be a finite set with $|A| = n$. The number of permutations of A is $n(n-1)\cdots 2 \cdot 1$. Since this number is constantly used in mathematics, $n!$ is the standard symbol for this number; i.e.,

$$n! = n(n-1)\cdots 2 \cdot 1.$$

An r**-subset** of A is a subset with r elements. An r-subset of A is also called an r**-combination** of A. The number of r-combinations of A is denoted by $\binom{n}{k}$, read "n choose r."

Proposition 5. *For non-negative integers n, k with $n \geq k$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. There are $\binom{n}{k}$ k-subsets of A. For each k-subset K of A, there are $k!$ ways to arrangement the elements of S. Thus

$$P(n,k) = \binom{n}{k} k!.$$

Note that $P(n,r) = \frac{n!}{(n-k)!}$. It follows that

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{k!(n-k)!}.$$

$\square$

Proposition 6. (a) $\binom{n}{r} = \binom{n}{n-r}$.

(b) $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Let $A = \{1, 2, \ldots, n\}$. We think of A as arry of boxes as follows

1	2	$\cdots$	n

Then an r-subset S may be considered as an array of boxes filled with 0 or 1, where a box is filled with 0 if the element is not in the set S and filled with 1 if the element is in the set S. For instance, let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The subset $\{2, 3, 5, 7, 8\}$ can be represented by

1	2	3	4	5	6	7	8
0	1	1	0	1	0	1	1

Corollary 7. *The number of words of* 0 *and* 1 *of length n with exactly r 1's and $(n-r)$ 0's is equal to $\binom{n}{r}$.*

Lecture 5

6.2 Binomial Theorem

Theorem 8 (Binomial Theorem)**.** *Let n be a positive integer, and let a, b be real numbers. Then*

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.$$

Proof. Since $(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{r}$, the expansion of the product is a sum of all 'words' of a and b with length n. The sum can be sorted into a sum from 0 to n by collecting the like terms with exaxtly the same number a's and the same number of b's. That is,

$$\begin{aligned}(a+b)^n &= \sum \{\text{words of } a \text{ and } b \text{ with length } n\} \\ &= \sum_{r=0}^{n} \{\text{words of } a \text{ and } b \text{ with exactly } r\ a\text{'s and exactly } (n-r)\ b\text{'s}\} \\ &= \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.\end{aligned}$$

□

Corollary 9. *For any positive integer n,*

$$\Big(1+x\Big)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Proposition 10. (a) *For integers r and n such that $0 \le r \le n$,*

$$\binom{n}{r} = \binom{n}{n-r}.$$

(b) *For integers r, n such that $0 \le r < n$,*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

6.3 Multinomial Coefficients

Let S be a finite set with n elements. Let $r_1, r_2, \ldots, r_k$ be nonnegative integers such that $n = r_1 + r_2 + \cdots + r_k$. We denote by

$$\binom{n}{r_1, r_2, \ldots, r_k}$$

the number of ways to partition S into a collection of ordered disjoint subsets $S_1, S_2, \ldots, S_k$ such that

$$|S_1| = r_1,\ |S_2| = r_2,\ \ldots,\ |S_k| = r_k.$$

Theorem 11.

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \cdots r_k!}.$$

Proof. Let S be a set with $|S| = n$. For each ordered partition $S_1, S_2, \ldots, S_k$ of S with $|S_1| = r_1$, $|S_2| = r_2$, …, $|S_k| = r_k$, the 1st subset S_1 has $r_1!$ permutations, the 2nd subset S_2 has $r_2!$ permutations, …, and the kth subset S_k has $r_k!$ permutations; each permutation of S can be obtained in this way. Hence

$$\binom{n}{r_1, r_2, \ldots, r_k} r_1! r_2! \cdots r_k! = n!.$$

□

Let S be a set. A multiset M over S is a collection of objects from S that the elements of S can be repeated; the repeated objects are indistinguishable. For instance, the collection $\{a, a, b, c, c, c\}$ is a multiset of 6 objects over the set $\{a, b, c\}$; of course it is also a multiset over the set $\{a, b, c, d\}$ with 0 copies of d. Let M be a multiset of n objects over S. If S has k elements and is ordered as $S = \{x_1, x_2, \ldots, x_k\}$, we say that a multiset M over S if of **type** $(r_1, r_2, \ldots, r_k)$ if M has r_1 copies of the 1st object x_1, r_2 copies of the 2nd object x_2, …, and r_k copies of the kth object x_k. For instance, $\{a, a, b, c, c, c\}$ is a multiset of type $(2, 1, 3)$ over $\{a, b, c\}$, but a multiset of type $(2, 1, 3, 0)$ over $\{a, b, c, d\}$.

Theorem 12. *Let M be a multiset of type $(r_1, r_2, \ldots, r_k)$ with $n = r_1 + r_2 + \cdots + r_k$. Then the number of permutations of M is*

$$\binom{n}{r_1, r_2, \ldots, r_k}.$$

In other words, this is the number of words of length n over a k-set, such that the 1st, the 2nd, …, and the kth objects appear exactly r_1, r_2, …, and r_k times, respectively.

Proof. Let $P(n; r_1, r_2, \ldots, r_k)$ denote the number of permutations of M. To figure out $P(n; r_1, r_2, \ldots, r_k)$, we may label the indistinguishable objects of type i in M by numbers $1, 2, \ldots, r_i$, where $1 \le i \le k$, so that M becomes a set N of n objects. There are $n!$ permutations of N.

On the other hand, the permutations of N can be obtained by labelling the objects in permutations of M. For each permutation of M, the objects of the type i can be labelled by $1, 2, \ldots, r_i$ in $r_i!$ ways, $1 \le i \le k$. Then each permutation of M produces $r_1! r_2! \cdots r_k!$ distinct permutations of N. It is clear that distinct permutations of M produce distinct permutations of N in this labelling. Thus

$$P(n; r_1, r_2, \ldots, r_k) r_1! r_2! \cdots r_k! = n!.$$

So we obtain the answer $P(n; r_1, r_2, \ldots, r_k) = \frac{n!}{r_1! r_2! \cdots r_k!}$. □

6.4 Multinomial Theorem

Theorem 13 (Multinomial Theorem)**.** *Let n be a positive integer, and let $x_1, x_2, \ldots, x_n$ be real numbers. Then*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{r_1+r_2+\cdots+r_k=n \\ r_1,r_2,\ldots,r_k \geq 0}} \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}.$$

Proof.

$$\begin{aligned}
(x_1 + x_2 + \cdots + x_k)^n &= \sum \Big(\text{words of length } n \text{ over } \{x_1, x_2, \ldots, x_k\} \Big) \\
&= \sum_{r_1+\cdots+r_k=n} \begin{pmatrix} \text{words of length } n \text{ over } \{x_1, \ldots, x_k\} \\ \text{with exactly } r_1 \ x_1\text{'s}, \ldots, r_k \ x_k\text{'s} \end{pmatrix} \\
&= \sum_{\substack{r_1+r_2+\cdots+r_k=n \\ r_1,r_2,\ldots,r_k \geq 0}} \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}.
\end{aligned}$$

$\square$